

On the numerical simulation of propagation of micro-level uncertainty for chaotic dynamic systems

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Abstract In this paper, a fine numerical algorithm, namely the “clean numerical simulation” (CNS), is proposed to accurately simulate the propagation of micro-level uncertainty of chaotic dynamic systems. The chaotic Hamiltonian Hénon-Heiles system for motion of stars orbiting in a plane about the galactic center is used as an example to show the validity of the CNS. It is found that, due to the sensitive dependence on initial condition (SDIC) of chaos, the micro-level inherent uncertainty of the position and velocity of a star transfers into the macroscopic randomness of motion. Thus, chaos is a bridge from the micro-level inherent uncertainty to the macroscopic randomness in nature. This provides us a new explanation to the SDIC of chaos from the physical viewpoint.

1 Introduction

Using high performance digit computers, a lots of complicated problems in science, finance and engineering have been solved with satisfied accuracy. However, there exist some problems which are still rather difficult to solve even by means of the most advanced computers. One of them is the propagation of micro-level uncertainty of chaotic dynamical systems.

It is well-known that all numerical simulations are not “clean”: there exist more or less numerical noises such as truncation error and round-off error, depending on numerical algorithms. In most cases, such kind of numerical noises are much larger than the micro-level uncertainty of dynamic systems under consideration, so that the micro-level uncertainty is completely lost in the numerical noise. This becomes more serious for chaotic dynamic systems, which have the sensitive dependence on initial conditions (SDIC), i.e. small difference of initial condition leads to great difference of numerical simulations of chaotic solution at large enough time. Thus, very fine numerical algorithms need be proposed so as to accurately simulate the propagation of micro-level uncertainty of chaotic dynamic systems. This is the motivation of this article.

Physically, many chaotic dynamic systems contain micro-level uncertainty in initial conditions. For example, let us consider the famous chaotic dynamic system governed by Lorenz equation, which is based on Navier-Stokes equation under the so-called continuum-assumption of fluid. In physics, any fluids are made of atoms or molecules, and thus *not* a continuum. So, the concepts in fluid mechanics such as velocity, density, pressure, as so on, are defined in the statistic meanings: strictly

speaking, all of them have statistic fluctuations and thus are *inherently* uncertain. Although these micro-level uncertainty might be much less than numerical noises of traditional numerical algorithms, they may propagate together with the chaotic solution and, due to the SDIC of chaos, have a great influence on chaotic solution at long enough time. Such kind of propagation of micro-level uncertainty has important physical meanings, and may deepen our understandings about the SDIC of chaotic dynamic system from physical and statistic view-points.

In this article, a kind of fine numerical algorithm, called “clean” numerical simulation (CNS), is proposed to accurately simulate propagation of micro-level uncertainty of chaotic dynamic systems. Here, the word “clean” means that the truncation error and round-off error are much less than the micro-level uncertainty so that the numerical noises can be neglected in a long enough interval of time for the propagation of uncertainty. A chaotic Hamiltonian system proposed by Hénon and Heiles [7] is used to show its validity. The basic ideas of the so-called clean numerical simulation (CNS) are given in §2, followed by the investigation of the micro-level uncertainty of the system in §3 and its propagation in §4 from statistical points of view. Conclusions and discussions are given in §5.

2 The numerical algorithm of the CNS

Hénon and Heiles [7] proposed a Hamiltonian system of equations

$$\ddot{x} = -x - 2xy, \tag{1}$$

$$\ddot{y} = -y - x^2 + y^2, \tag{2}$$

to approximate the motion of stars orbiting in a plane about the galactic center, where the dot denotes the differentiation with respect to the time. Its solution is chaotic for some initial conditions, such as

$$x(0) = 0.56, y(0) = 0, \dot{x}(0) = 0, \dot{y}(0) = 0,$$

as mentioned by Sprott [16]. Without loss of generality, let us use this chaotic system to describe the basic ideas of the CNS and to show its validity.

It is well-known [3, 5, 8, 10, 11, 15, 16, 18] that chaotic dynamic systems have the sensitive dependence on initial conditions (SDIC), i.e. small difference of initial conditions lead to great difference of numerical simulations for large time, so that long-term prediction of chaos is impossible. Since the traditional numerical simulations contain the unavoidable truncation and round-off errors at each time-step, which are not negligible, all traditional numerical simulations of chaos are mixed with these numerical noises and thus are not “clean”. Because these numerical noises are generally much larger than the micro-level inherent uncertainty of physical variables, the propagation of such kind of physical uncertainty of chaotic dynamic systems has never been studied in details.

In order to gain reliable chaotic solutions in a long enough interval of time, Liao [9] developed a numerical technique with extremely high precision, called here the “clean numerical simulation” (CNS). Using the computer algebra system Mathematica with the 400th-order Taylor expansion for continuous functions and data in accuracy of 800-digit precision, Liao [9] gained, for the first time, the “clean” numerical results of chaotic solution of Lorenz equation in a long interval $0 \leq t \leq 1000$ LTU (Lorenz time unit) with negligible truncation and round-off error. The basic ideas of the CNS are simple and straightforward. Since the order of Taylor expansion is very high, the corresponding truncation error is rather small. Besides, expressing all data in the accuracy of large-number digit precision, the small enough round-off error is guaranteed. Thus, as long as the order of Taylor expansion is high enough and the digit-number of data is long enough, both of the truncation and round-off errors can be much smaller than the micro-level uncertainty so that the propagation of micro-level uncertainty can be simulated accurately in a long enough interval of time. Currently, Liao’s “clean” chaotic solution [9] of Lorenz equation is confirmed by Wang et al [17], who used parallel computation with the multiple precision (MP) library: they gained reliable chaotic solution up to 2500 LTU by means of the 1000th-order Taylor expansion and data in the accuracy of 2100-digit precision, and their result agrees well with Liao’s one [9] in $0 \leq t \leq 1000$ LTU. This kind of reliable “clean” chaotic solutions and especially the CNS provide us a way to accurately investigate the propagation of micro-level uncertainty of chaotic dynamic systems, which may give us some new explanations to the essence of SDIC and the “butterfly effect” from the *physical* and *statistic* points of view, as shown below.

As mentioned above, the basic ideas of the CNS are easy to understand. The first step of the CNS is to give the high-order Taylor formulas for numerical simulation. Let (x_n, y_n) and (\dot{x}_n, \dot{y}_n) denote the position and velocity at the time $t_n = n\Delta t$, where Δt is a constant for time-step. Write

$$x(t) = \sum_{n=0}^{+\infty} a_n (t - t_n)^n, \quad y(t) = \sum_{n=0}^{+\infty} b_n (t - t_n)^n, \quad t_n \leq t \leq t_n + \Delta t,$$

where $a_0 = x_n, b_0 = y_n, a_1 = \dot{x}_n, b_1 = \dot{y}_n$ are known. Substituting them into the original governing equations of the Hénon and Heiles system [7] and equaling the like power of $\delta t = t - t_n$, we have the recursion formula

$$a_{n+2} = -\frac{a_n + 2 \sum_{k=0}^n a_k b_{n-k}}{(n+1)(n+2)}, \quad (3)$$

$$b_{n+2} = -\frac{b_n + \sum_{k=0}^n (a_k a_{n-k} - b_k b_{n-k})}{(n+1)(n+2)} \quad (4)$$

for $n \geq 0$. Then, we have the M th-order Taylor approximation

$$x_{n+1} \approx \sum_{k=0}^M a_k (\Delta t)^k, \quad y_{n+1} \approx \sum_{k=0}^M b_k (\Delta t)^k \quad (5)$$

and

$$\dot{x}_{n+1} \approx \sum_{k=0}^{M-1} (k+1)a_{k+1} (\Delta t)^k, \quad (6)$$

$$\dot{y}_{n+1} \approx \sum_{k=0}^{M-1} (k+1)b_{k+1} (\Delta t)^k \quad (7)$$

at the time $t_{n+1} = (n+1)\Delta t$. Besides, all data must be expressed in a large enough number of digit precision. In this article, for the sake of simplicity, all data are expressed in the accuracy of $2M$ -digit precision (we use the computer algebra system Mathematica here), as long as M , the order of Taylor expansion, is large enough. In this way, one gains the high-order numerical simulations of $x(t)$ and $y(t)$ step by step, with small enough truncation and round-off errors.

Let T_c denote the maximum time for given M and Δt , up to which the chaotic solution is “clean”, i.e. without observable influence of the round-off and truncation errors, and thus is reliable. The second step of the CNS is to determine a good approximation of T_c , an important characteristic length-scale of time for the CNS. Without loss of generality, we use here the M th-order Taylor formula with $\Delta t = 1/10$ and the data in accuracy of $(2M)$ -digit precision. It is found that

$$T_c \approx 32 + 32M,$$

and

$$T_c \approx 21 - 23 \log_{10} |\sigma|,$$

i.e.

$$|\sigma| \leq 10^{0.875 - 0.044T_c},$$

where σ is the standard deviation of physical fluctuation of position and velocity in initial conditions. For examples of gaining such kind of formulas, please refer to Liao [9]. Thus, in order to gain “clean” reliable chaotic solution of the Hénon and Heiles system [7] in the interval $0 \leq t \leq 2000$, we should use at least the 70th-order Taylor expansion (with $\Delta t = 1/10$) and the data in accuracy of 140-digit precision.

Note that the numerical noises of each simulation given by the CNS are negligible in the interval $0 \leq t \leq 2000$, so that we can investigate the propagation of the inherent micro-level uncertainty of the position and velocity at $t = 0$, which are much larger than the numerical noise, as shown below.

3 The micro-level uncertainty

The kinetic status of a star is determined by its position and velocity. Traditionally, it is believed that the kinetic status of a star is *inherently* exact and the uncertainty of position and velocity come from the imperfect measure equipments. However, as

pointed out below, this traditional idea is suspectable: the position and thus the velocity of a star are inherently uncertain. Thus, let us consider the initial conditions

$$x(0) = x_0 + \tilde{x}_0, \quad y(0) = y_0 + \tilde{y}_0, \quad \dot{x}(0) = u_0 + \tilde{u}_0, \quad \dot{y}(0) = v_0 + \tilde{v}_0,$$

where x_0, y_0, u_0, v_0 are *observable* values of the initial position and velocity of a star, and $\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0$ are the corresponding *micro-level* uncertain ones, respectively. Assume that (x_0, y_0, u_0, v_0) is exactly defined and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ is in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$. The reasons for the above assumptions are given below.

It is a common belief of the scientific society that the microscopic phenomenon are essentially uncertain and random. To show this point, let us consider some typical length scales of microscopic phenomenon widely used in modern physics. For example, Bohr radius

$$r = \frac{\hbar^2}{m_e e^2} \approx 5.2917720859(36) \times 10^{-11} \text{ (m)}$$

is the approximate size of a hydrogen atom, where \hbar is a reduced Planck's constant, m_e is the electron mass, and e is the elementary charge, respectively. Besides, Compton wavelength $L_c = \hbar/(mc)$ is a quantum mechanical property of a particle, i.e. the wavelength of a photon whose energy is the same as the rest-mass energy of the particle, where m is the rest-mass of the particle and c is the speed of light. It is the length scale at which quantum field theory becomes important. The value for the Compton wavelength of the electron is

$$L_c \approx 2.4263102175(33) \times 10^{-12} \text{ (m)}.$$

In addition, the Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616252(81) \times 10^{-35} \text{ (m)}, \quad (8)$$

where c is the speed of light in a vacuum, G is the gravitational constant, and \hbar is the reduced Planck constant, is the length scale at which quantum mechanics, gravity and relativity all interact very strongly. Especially, according to string theory [14], the Planck length is the order of magnitude of the oscillating strings that form elementary particles, and *shorter length do not make physical senses*. Besides, in some forms of quantum gravity, it becomes *impossible to determine* the difference between two locations less than one Planck length apart. Therefore, in the accuracy of the Planck length level, the position of a star is inherently uncertain, so is its velocity. Note that this kind of uncertainty is *inherent* and has nothing to do with the Heisenberg uncertainty principle [6].

On the other hand, according to de Broglie [2], any body has the so-called wave-particle duality, and the length of the so-called de Broglie wave is given by

$$\lambda = \frac{h}{mv} \sqrt{1 - \left(\frac{v}{c}\right)^2}, \quad (9)$$

where m is the rest mass, v denotes the velocity of the body, c is the speed of light, h is the Planck's constant, respectively. Note that, the de Broglie's wave of a body has non-zero amplitude, meaning that the position is uncertain: it could be almost *anywhere* along the wave packet. Thus, according to the de Broglie's wave-particle duality, the position of a body is *inherent* uncertain.

Therefore, it is reasonable for us to assume that the micro-level inherent fluctuation of position of a star shorter than the Planck length l_p is essentially uncertain and/or random.

To gain the dimensionless Planck length l_p , we use the dimeter of Milky Way Galaxy as the characteristic length, say, $d_M \approx 10^5$ (light year) $\approx 9 \times 10^{20}$ (m). Obviously, $l_p/d_M \approx 1.8 \times 10^{-56}$ is a rather small dimensionless number. As mentioned above, two (dimensionless) positions shorter than 10^{-56} do not make physical senses. Thus, it is reasonable to assume the existence of the inherent uncertainty of the position and velocity of a star in the normal distribution with zero mean and the micro-level standard deviation 10^{-60} , which is added to the *observable* values (x_0, y_0, u_0, v_0) of the initial conditions.

Generally speaking, the number 10^{-60} is indeed rather small. The propagation of such a tiny inherent fluctuation of physical variables with chaos has never been investigated, since the numerical noises (i.e. truncation and round-off errors) of traditional numerical simulations of chaos are much larger than the uncertainty in general. However, by means of the CNS with the 70th-order Taylor expansion and the data in accuracy of 140-digit precision, such kind of propagation can be accurately studied, for the first time, because 10^{-60} is a hugh number compared to 10^{-140} : the number $10^{-140}/10^{-60} = 10^{80}$ is much greater even than d_M/l_p , the ratio of the dimeter of Milky Way Galaxy to the Planck length!

4 Statistic property of chaos

Without loss of generality, let us consider the case of the observable values

$$x_0 = 14/25, y_0 = 0, u_0 = 0, v_0 = 0$$

of the initial conditions, corresponding to a chaotic motion [15]. Let

$$\langle x(t) \rangle = \frac{1}{N} \sum_{i=1}^N x_i(t), \quad \sigma_x(t) = \sqrt{\frac{1}{N-1} \sum_{i=1}^N [x_i(t) - \langle x(t) \rangle]^2}$$

denote the sample mean and unbiased estimate of standard deviation of $x(t)$, respectively, where $x_i(t)$ is the i th sample given by CNS for a tiny random term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ with the micro-level deviation $\sigma = 10^{-60}$ in the initial condition, and $N = 10^4$ is the number of total samples gained by the CNS mentioned above.

The standard deviations $\sigma_x(t)$ and $\sigma_y(t)$ of $x(t), y(t)$ are as shown in Figs. 1 and 2, respectively. Note that there exists an interval $0 \leq t \leq T_d$ with $T_d \approx 1000$, in

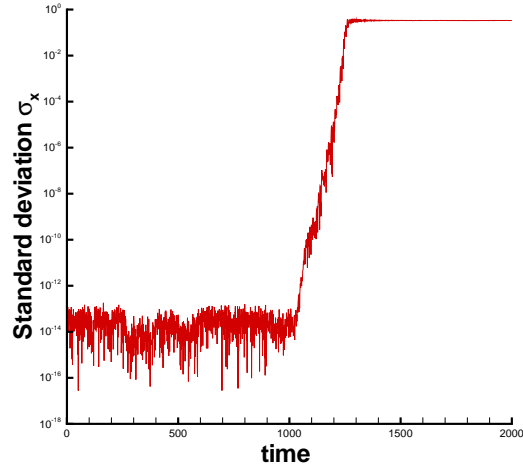


Figure 1: The standard deviation σ_x of x in case of $x_0 = 14/25, y_0 = 0, u_0 = 0, v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

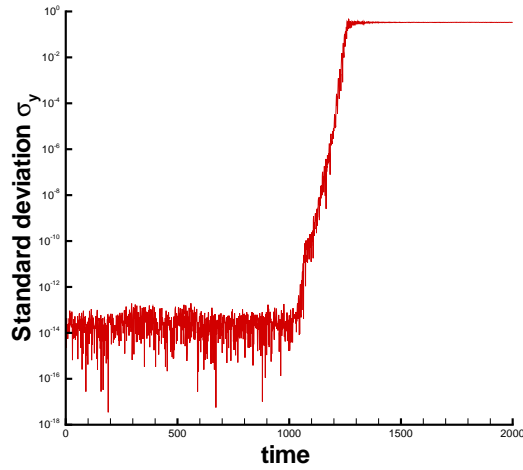


Figure 2: The standard deviation σ_y of y in case of $x_0 = 14/25, y_0 = 0, u_0 = 0, v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

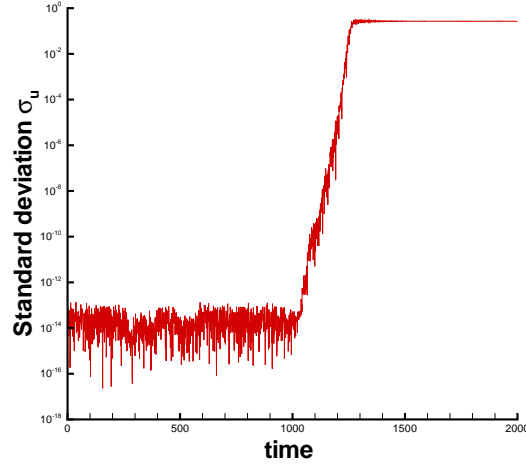


Figure 3: The standard deviation σ_u of \dot{x} in case of $x_0 = 14/25, y_0 = 0, u_0 = 0, v_0 = 0$ and the uncertain term $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the normal distribution with zero mean and a micro-level deviation $\sigma = 10^{-60}$.

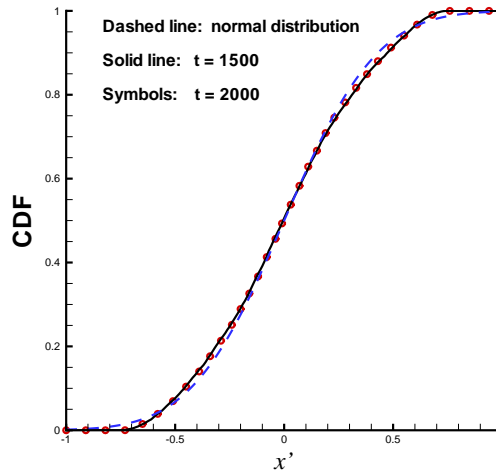


Figure 4: The CDF of x' , compared to the normal distribution (dashed line) with zero mean and the standard deviation of x' at $t = 2000$. Solid line: CDF of x' at $t = 1500$; symbols: CDF of x' at $t = 2000$.

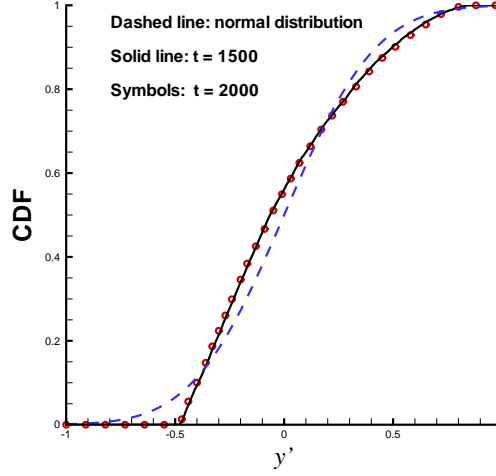


Figure 5: The CDF of y' , compared to the normal distribution (dashed line) with zero mean and the standard deviation of y' at $t = 2000$. Solid line: CDF of y' at $t = 1500$; symbols: CDF of y' at $t = 2000$.

which $\sigma_x(t)$ and $\sigma_y(t)$ are in the level of 10^{-14} so that one can accurately predict the position (x, y) of a star, even if the corresponding motion is chaotic and the initial condition contains uncertainty. Similarly, the velocity of the star can be also precisely predicted in $0 \leq t \leq T_d$, as shown in Fig. 3 for the standard deviation $\sigma_u(t)$ of $\dot{x}(t)$. Thus, when $0 \leq t \leq T_d$, the behavior of the system looks like “deterministic” and predictable, even from the statistic viewpoint. When $t > T_d$, the standard deviations of the position and velocity begin to increase rapidly, and thus the system becomes random obviously: the position (x, y) and velocity (\dot{x}, \dot{y}) of the star are strongly dependent upon their tiny inherent random terms $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ of the initial condition in the micro-level accuracy of 10^{-60} . In other words, due to the SDIC of chaos, the unobservable micro-level inherent uncertainty of the position and velocity of a star transfers into the macroscopic randomness of the motion. So, chaos is a bridge from the micro-level uncertainty to macroscopic randomness! Therefore, the micro-level inherent uncertainty of the position and velocity is an origin of the microscopic randomness of motion of stars in our universe. This provides us a new physical explanation for the SDIC of chaos. For this reason, each “big bang” [12] will crease a completely different universe!

Besides, it is found that the standard deviations of the position and velocity become almost stationary when $t > T_s$, where $T_s \approx 1300$, as shown in Figs. 1 to 3. Thus, when $T_d < t < T_s$, the system is in the transition process from the “deterministic” behavior to the stationary randomness. It is interesting that the stationary standard deviations of $x(t)$ and $y(t)$ are about $1/3$, and their stationary means $\langle x \rangle$ and $\langle y \rangle$ are close to zero. It means that, due to SDIC of chaos and the micro-level inherent uncertainty of position and velocity, a star orbiting in a plane about the galactic center could be almost *everywhere* in the galaxy at a given time $t > T_s$.

Write the fluctuations $x' = x - \langle x \rangle$ and $y' = y - \langle y \rangle$. The stationary cumulative distribution functions (CDF) of x', y' are almost independent of time, as shown in Figs. 4 and 5. Besides, the stationary CDF of the fluctuation x' is rather close to the normal distribution with zero mean and the standard deviation of x' , as shown in Fig. 4. But, the stationary CDF of the fluctuation y' is obviously different from the normal distribution, as shown in Fig. 5.

Similarly, we investigate the influence of the observable values (x_0, y_0, u_0, v_0) and the standard deviation σ of the uncertain terms $(\tilde{x}_0, \tilde{y}_0, \tilde{u}_0, \tilde{v}_0)$ in the initial condition. It is found that T_d decreases exponentially with respect to σ . Besides, the stationary means and standard deviations of x, y, \dot{x}, \dot{y} , and the CDFs of x' and y' , are independent of the observable values (x_0, y_0, u_0, v_0) . Thus, when $t > T_s$, all observable information of the initial condition are lost completely. In other words, when $t > T_s$, the asymmetry of time breaks down so that the time has a one-way direction, i.e. the arrow of time. So, statistically, the Hénon-Heiles system has two completely different dynamic behaviors before and after T_d : it looks like “deterministic” and predictable without time’s arrow when $t \leq T_d$, but thereafter rapidly becomes obviously random with arrow of time.

5 Conclusions and discussions

In this paper, a fine numerical algorithm, namely the “clean numerical simulation” (CNS), is proposed to accurately simulate the propagation of micro-level uncertainty of chaotic dynamic systems. The chaotic Hénon-Heiles system is used as an example to show the validity of the CNS.

Based on the assumption that the micro-level inherent fluctuation of position of any a body is essentially uncertain and/or random in the accuracy of micro-level shorter than the Planck length l_p , our CNS simulations indicate that, due to the SDIC of chaos, the micro-level inherent uncertainty of position and velocity of a star transfers into macroscopic random motion. Thus, the macroscopic randomness of motion of stars has a close relationship with the microscopic uncertainty of position and velocity: chaos is a bridge from the micro-level uncertainty to macroscopic randomness. This kind of relationship gives an explanation and origin of the random motion of stars in the universe. It also provides us a new physical explanation to the SDIC of chaos. Since chaos and the microscopic uncertainty of position and velocity of a star are unavailable, each “big bang” [12] will create a completely different universe.

Note that the Planck length l_p in the assumption is not the key point. Replacing l_p by $10^{-30} \times l_p$, we gain qualitatively the same conclusions. Besides, our assumption is consistent with the Heisenberg uncertainty principle [6]

$$\Delta x \Delta(mv) \geq h,$$

where h is the Planck constant, which indicates that

$$\Delta x \geq \frac{h}{m\Delta v} \geq \frac{h}{mc},$$

say, it is impossible to measure a position more accurate than $h/(mc)$, since the uncertainty of velocity Δv is not greater than the light speed, according to the relativity theory [4]. So, according to the Heisenberg uncertainty principle [6], it is impossible to measure the position of a star (with the mass m) more accurate than $h/(mc)$. Thus, since m and c are finite, it is impossible to measure a position with an *arbitrary* degree of accuracy and certainty, *forever*. So, considering the string theory [14] and de Broglie's wave-particle duality, we have many reasons to make the assumption used in this article: the position of a star is *inherently* uncertain, and such kind of uncertainty has nothing to do with measurement done by human being.

Strictly speaking, the micro-level uncertain fluctuation of the position and velocity of a star exists not only at $t = 0$, but also in the whole interval of time. Adding the micro-level fluctuation to the calculated results at each time step, we gain the *same* stationary means, standard deviations and CDFs of x, y , and the *same* conclusion that the inherent micro-level uncertainty of the position and velocity transfers into the macroscopic randomness of motion of a star.

Note also that the Hénon-Heiles system is a model for a star orbiting in a plane about the galactic center. Any a mathematical model is more or less a kind of approximation of physical phenomena. However, one can gain the qualitatively same conclusions, even if the Hénon-Heiles system is replaced by other better models someday, as long as their solutions are chaotic in some cases. Due to the Newton's gravitational law, the motion of stars are governed by nonlinear dynamic equations so that chaotic motions of stars are unavailable [1]. Thus, if the equations of the Hénon-Heiles system are replaced by those for the famous three-body problem [13], one can gain the qualitatively same conclusions. So, our conclusions reported in this article have general meaning.

Traditionally, the deterministic means that the evolution of solutions is fully determined by initial conditions *without* random or uncertain elements involved. Strictly speaking, such kind of deterministic models do not exist at all for physical phenomena, since no one can ensure that there exist no random and uncertain elements in initial/boundary conditions. So, strictly speaking, the Hénon-Heiles system is *not* a deterministic model even from the traditional viewpoint. Similarly, the three-body problem is also *not* a deterministic one, since the *inherent* uncertainty of position and velocity in its initial condition is unavailable, as mentioned above. Thus, strictly speaking, the traditional "deterministic chaos" does not exist for models having physical meanings: only a few pure mathematical models without physical meanings are "deterministic chaos". In this article, we illustrate that a nonlinear dynamic system with chaos can be predicted with a rather high-level accuracy in a long interval of time and thus behaves like "deterministic" and predictable, even from the statistic viewpoint. This might provide us a new definition of the "deterministic", which is more practical than the traditional ones.

It is traditionally believed that the long-term prediction of chaos is impossible, mainly due to the impossibility of the perfect *measurement* of initial conditions with an *arbitrary* degree of accuracy. This is the traditional explanation to the SDIC of

chaos. Here, we provide a new explanation for the SDIC of chaos from the physical and statistic viewpoint: the observable initial conditions are assumed to be exact, but the *inherent* micro-level uncertainty of the physical variables are propagated into macroscopic randomness. Different from the traditional explanation of the SDIC, which focuses on the *measurement*, the new explanation emphasizes the *inherent* micro-level uncertainty of physical variables and its propagation with chaos. Besides, it should be emphasized that such micro-level uncertainty of physical variables was completely inundated with the numerical “noises” of the traditional numerical methods, and thus have never been studied in details. This shows the validity and potential of the CNS to precisely simulate complex physical phenomena with the SDIC, such as weather prediction and turbulence.

Finally, it should be emphasized that, the macroscopic randomness is traditionally believed to have *no* relationships with the micro-level uncertainty. However, our work illustrates that they have a *close* relationship: chaos is a bridge from the micro-level inherent uncertainty to macroscopic randomness!

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